Advanced Topics on Weighted Tree Automata

Exercise 1 (auMSO to rec. step function)
Consider the semiring \((\mathbb{N}, +, \cdot, 0, 1)\) and let \(\Sigma\) be a ranked alphabet s.t. \(\{\gamma^{(1)}, \sigma^{(2)}\} \subseteq \Sigma\). Presuming the almost unambiguous MSO(\(\Sigma, \mathbb{N}\))-formula
\[
\varphi = (\text{label}_\sigma(x) \lor \text{label}_\gamma(x)) \land (3 \lor \text{edge}_2(x, y)),
\]
give MSO(\(\Sigma\))-formulae \(\psi_i\) and numbers \(n_i \in \mathbb{N}, i \in [k]\), such that
\[
\llbracket \varphi \rrbracket_\mathcal{V} = \sum_{i=1}^{k} n_i \llbracket \psi_i \rrbracket
\]
for every \(\mathcal{V} \supseteq \text{Free}(\psi)\).

Exercise 2 (srMSO to MExp)
Consider the semiring \((\mathbb{N}, +, \cdot, 0, 1)\) and let \(\sigma, \gamma \in \Sigma\) be arbitrary symbols from a presupposed ranked alphabet \(\Sigma\). Compute, according to Fülöp, Stüber, and Vogler (2010, Lem. 5.10), the M-expression \(t(\varphi)\) over \(\Sigma\) and \(\mathcal{A}_\mathbb{N}\), where \(\varphi\) is the srMSO-formula
\[
\varphi = \forall x. (\text{label}_\sigma(x) \lor \text{label}_\gamma(x)) \lor \exists x. (\text{label}_\sigma(x) \land 3).
\]

Exercise 3 (MExp to srMSO)
Let \(\Sigma = \{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\}\) and assume the M-expression
\[
e = \sum_{\mathcal{V}} \left( \neg \forall x. (\text{label}_\sigma(x) \lor \text{label}_\beta(x)) \right) \triangleright H(\omega)
\]
over \(\Sigma\) and \(\mathcal{A}_\mathbb{N}\), where \(\omega\) is the \(\Sigma_{\{\mathcal{V}\}}\)-family of operations defined by
\[
\omega(\tau, \emptyset) = \text{mul}_1 \quad \omega(\tau, \{\mathcal{V}\}) = \text{mul}_3
\]
for every \(\tau \in \Sigma\). Compute the syntactically restricted MSO(\(\Sigma, \mathbb{N}\))-formula \(t(e)\) according to Lem. 5.12 (ibid.).

References