Advanced Topics on Weighted Tree Automata

Exercise 1 (wmta vs. automata over valuation monoids)
Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. Construct a wmta $M$ over a suitable M-monoid $\mathcal{A}$ such that, for every $\xi \in T_\Sigma$,

$$\llbracket M \rrbracket (\xi) = \frac{|\text{pos}_\alpha(\xi)|}{|\text{pos}(\xi)|},$$

that is, the automaton computes the ratio of the number of leaves in $\xi$ to the size of $\xi$.

Exercise 2 (Semantics of M-expressions)
Let us assume a ranked alphabet $\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$. Analyze the semantics $\llbracket e_i \rrbracket$ of the following M-expressions $e_i \in \text{MExp}(\Sigma, \mathcal{A}_i)$, $i \in \mathbb{N}$.

- $e_1 = H(\omega)$, with $\omega_\alpha() = 1$, $\omega_\gamma(x) = x$, $\omega_\sigma(x_1, x_2) = 2 \cdot x_1 \cdot x_2$, and $\mathcal{A}_1 = (\mathbb{N}, +, 0, \Omega_1)$
- $e_2 = \sum_{k \in \mathbb{N}} (\text{label}_\sigma(x) \land \exists y. (\text{edge}_2(x, y) \land \text{label}_\alpha(y))) \triangleright H(\theta)$, where $\theta_\gamma(x_1, \ldots, x_k) = 1$ for every $\tau \in \Sigma^{(k)}$, $k \in \mathbb{N}$, and $\mathcal{A}_2 = (\mathbb{N}, +, 0, \Omega_2)$.

- $e_3 = \sum_x \varphi \triangleright H(\eta)$, with

\[
\varphi(X) = \forall x. x \in X \rightarrow (\text{label}_\sigma(x) \lor \text{label}_\gamma(x)) \\
\land \forall y. y \in X \rightarrow (\text{edge}(x, y) \rightarrow \text{label}_\sigma(x) \land \text{label}_\gamma(y) \lor \text{label}_\sigma(y)) \\
\land \exists Y. (\psi(Y, x, y) \lor \psi(Y, y, x)) \land Y \subseteq X \\
\psi(X, x, y) = x \in X \land y \in X \\
\land \forall z. z \in X \land \neg z = x \rightarrow \exists u. (u \in X \land \text{edge}(z, u)) \\
\land \forall z. z \in X \land \neg z = y \rightarrow \exists u. (u \in X \land \text{edge}(u, z)),
\]

as well as

\[
\eta_{(\alpha, \theta)}() = 0 \quad \eta_{(\alpha, [\chi])}() = 0 \\
\eta_{(\gamma, \delta)}(x) = x \quad \eta_{(\gamma, [\chi])}(x) = x + 1 \\
\eta_{(\sigma, \beta)}(x_1, x_2) = x_1 + x_2 \quad \eta_{(\sigma, [\chi])}(x_1, x_2) = x_1 + x_2 + 1
\]

and $\mathcal{A}_3 = (\mathbb{N} \cup \{-\infty\}, \text{max}, -\infty, \Omega_3)$.

The sets of operations $\Omega_i$, $i \in \mathbb{N}$, are chosen suitably, i.e. they contain the operations used in the respective automata, as well as the functions $0^{(k)}$, $k \in \mathbb{N}$.

Exercise 3 (ubal)
In Example 3.6 (Fülöp, Stüber, and Vogler, 2010), an M-expression

$$e = \sum_i \sum_j \sum_{z_1} \sum_{z_2} \varphi(x, Y, Z_1, Z_2) \triangleright H(\omega_\delta \mid \delta \in \Sigma_\gamma)$$

was given, which defines the tree series ubal (cf. Ex. 2.4). Complete the definition of $e$ with the necessary MSO formula $\varphi(x, Y, Z_1, Z_2)$.
Exercise 4 (Absorption necessary for Theorem 4.1)
Presume $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$. Show that there is an $M$-monoid $\mathcal{A}$ that is not absorptive such that $\text{Rec}(\Sigma, \mathcal{A})$ is not closed under summation, i.e., there are weighted tree languages $r_1, r_2 \in \text{Rec}(\Sigma, \mathcal{A})$ such that $r_1 + r_2 \notin \text{Rec}(\Sigma, \mathcal{A})$.

References